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Approximate Bayesian Estimates of Weibull Parameters with Lindley's Method

(Anggaran Pengamiran Bayesian untuk Parameter Weibull Menggunakan Kaedah Lindley's)

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ABSTRACT

One of the most important lifetime distributions that is used for modelling and analysing data in clinical, life sciences and engineering is the Weibull distribution. The main objective of this paper was to determine the best estimator for the two-parameter Weibull distribution. The methods under consideration are the frequentist maximum likelihood estimator, least square regression estimator and the Bayesian estimator by using two loss functions, which are squared error and linear exponential. Lindley approximation is used to obtain the Bayes estimates. Comparisons are made through simulation study to determine the performance of these methods. Based on the results obtained from this simulation study the Bayesian approach used in estimating the Weibull parameters under linear exponential loss function is found to be superior as compared to the conventional maximum likelihood and least squared methods.

Keywords: Bayesian; least squarer; maximum Likelihood; squared error and linear exponential loss functions

ABSTRAK

Salah satu taburan jangka hayat yang sangat penting yang sering digunakan dalam pemodelan dan analisis data klinikal, sains hayat dan kejuruteraan adalah taburan Weibull. Objektif utama kertas ini adalah untuk menentukan penganggar yang paling baik bagi taburan Weibull dua-parameter. Kaedah yang dipertimbangkan adalah anggaran kebolehjadian maksimum, anggaran regresi kuasa dua terkecil dan anggaran Bayes menggunakan dua fungsi, iaitu ralat kuasa dua dan fungsi, linear eksponen. Pengamiran Lindley digunakan untuk memperoleh anggaran Bayes. Perbandingan dijalankan melalui simulasi untuk menentukan prestasi kaedah. Hasil yang diperoleh daripada kajian simulasi menunjukkan pendekatan Bayes dalam menganggar parameter Weibull dengan fungsi linear eksponen didapati lebih baik jika dibandingkan dengan kaedah konvensional kebolehjadiaan maksimum dan kaedah kuasa dua terkecil berdasarkan nilai ralat kuasa dua min.

Kata kunci: Bayesian; fungsi ralat kuasa dua dan linear eksponen; kebolehjadiaan maksimum; regresi kuasa dua terkecil

INTRODUCTION

Weibull distribution is one of the most widely used models for failure time in both lifetime and reliability analysis. It was shown to be very useful for modeling and analyzing life time data in the applied and engineering sciences (Lawless 1982).

The primary advantage of Weibull analysis as stated by Abernethy (2006) is its ability to provide accurate failure analysis and failure forecasts with extremely small samples. With Weibull, solutions are possible at the earliest indications of a problem without having to pursue farther. Small samples also allow cost effective component testing.

Maximum likelihood estimator (MLE) is quite efficient and very popular both in literature and practice as compared to least square estimator. Al Omari & Ibrahim, 2011 conducted a study on Bayesian survival estimator for Weibull distribution with censored data. (Pandey et al. 2011) compared Bayesian and maximum likelihood estimation of the scale parameter of Weibull with known shape parameter under LINEX and Syuan-Rong & Shuo-Jye (2011) considered Bayesian estimation and prediction for Weibull model with progressive censoring. Similar work can be seen in (Al-Athari 2011; Zellner 1986).

In this study consideration is given to the estimation of the two-parameter Weibull distribution by using maximum likelihood estimation (MLE), least square estimation (LSE) and Bayesian estimation under two loss functions, namely squared error loss also known as quadratic loss function and linear exponential (LINEX) loss function and under the assumption of a bivariate non-informative prior. The quadratic loss function is the loss incurred by adopting action a when the true state of nature is q and is given by:

$$L(a, \theta) = (a-\theta)^2.$$

According to Preda et al. 2010, it is remarkable that most of the Bayesian inference procedures have been developed with the usual quadratic loss function, which by its nature is symmetric thereby associating equal importance to the losses be it overestimation or underestimation, which is impractical in real life situations since in the estimation of reliability and failure rate functions, an overestimation may be more costly than underestimation or vice versa.

The linear exponential loss function which was introduced by Varian 1975 is given as

$$L(\Delta) = e^{a\Delta} - a\Delta - 1;\ a \neq 0,$$

where a determines the slope of the loss function such that when $a > 0$, there is a penalized overestimation and when $a < 0$, we have a penalized underestimation but in a situation where $a \cong 0$ then the linex loss is almost symmetric and approaches the quadratic loss function.

The rest of the study is organized as follows. In the next section, maximum likelihood estimators of the parameters are introduced, next is least square regression method and then followed by Bayesian estimation approach. Simulation study is carried out to investigate the performance of the estimators. This is followed by results and finally conclusion is given based on the simulation study.

Materials and Methods

Maximum likelihood estimation: Let $t_1; t_2; \dots t_n$ be a random sample of size *n* that follows Weibull distribution. The two parameter Weibull probability density function (pdf) and the cumulative distribution function (CDF) are

$$f(t.\alpha, \beta) = \alpha\beta t^{\beta-1}\exp(-\alpha t^{\beta}). \quad (1)$$

$$F(t, \alpha, \beta) = 1 - \exp(-\alpha t^{\beta}). \quad (2)$$

The likelihood function is

$$L(t_i,\alpha,\beta) = \prod_{i=1}^{n} f(t_i,\alpha,\beta) \quad (3)$$

With

$$L(t_i,\alpha,\beta) = f(t_1, t_2,\dots, t_n : \alpha,\beta) = \prod_{i=1}^{n}\left(\alpha\beta_{t_i}^{\beta-1}\exp\left(-\alpha t_i^{\beta}\right)\right) \quad (4)$$

Taking the natural logarithm through and equating to zero (0), we have

$$\ell = n\ln\alpha + n\ln\beta + (\beta-1)\sum_{i=1}^{n}\ln(t_i) - \alpha\sum(t_i)^{\beta} = 0 \quad (5)$$

Differentiating (5) with respect to alpha, we have

$$\hat{\alpha} = \left[\frac{n}{\sum_{i=1}^{n}(t_i)^{\beta}}\right] \quad (6)$$

Differentiating (5) again but with respect to b the shape parameter, we obtain

$$\frac{n}{\beta} + \sum_{i=1}^{n}\ln(t_i) - \hat{\alpha}\sum_{i=1}^{n}(t_i)^{\beta}\ln(t_i) = 0 \quad (7)$$

Substituting α from (6) into (7), we have

$$\frac{n}{\beta} + \sum_{i=1}^{n}\ln(t_i) - \left[n\frac{\sum_{i=1}^{n}(t_i)^{\beta}\ln(t_i)}{\sum_{i=1}^{n}(t_i)^{\beta}}\right] = 0 \quad (8)$$

$\hat{\beta}$ can be obtained from (8) iteratively by using Newton-Raphson method after which $\hat{\alpha}$ can be determined.

Least square regression: According to Gibbons & Vance 1981, if available data are plotted on Weibull probability paper (WPP) and the order of observations which are $t_1, \dots, t_n$ are represented on the abscissa against some plotting rule which estimates the CDF $F(t_i)$, then the failure time that has been observed will be converted by the probability paper to $\ln(t_i)$ and t to $\ln[-\ln(1-F(t_i))]$. Since $\ln[-\ln(1-F(t_i))]$ is random, then the sum of squares in the vertical direction can be minimized to obtain the estimates of the parameters by employing the following.

Taking natural log twice of (2) we have

$$\ln[-\ln(1-F(t_i))] = \ln(\alpha) + \beta\ln(t_i) \quad (9)$$

Let

$$y_i = \ln[-\ln(1-F(t_i))],\ a = \ln(\alpha),\ x_i = \ln(t_i),\ b = \beta.$$

Then a linear equation of the form $y_i = b_{xi} + a$ is obtained.

According to (Zhang et al. 2007),

$$\hat{\beta} = \left\{\frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right\} \quad (10)$$

$$\hat{\alpha} = \left\{\exp\left[-\left(\bar{y}/\beta - \bar{x}\right)\right]\right\} \quad (11)$$

substituting y_i and x_i as given above into (10) and (11) we obtain the following equations:

$$\hat{\beta} = \left\{\frac{n\sum_{i=1}^{n}\ln(t_i)\ln\sum_{i=1}^{n}\ln\left[-\ln\left(1-F(t_i)\right)\right] - \sum_{i=1}^{n}\ln(t_i)\sum_{i=1}^{n}\ln\left[-\ln(1-F)(t_i)\right]}{n\sum_{i=1}^{n}\ln(t_i)^2 - \left[\sum_{i=1}^{n}\ln(t_i)\right]^2}\right\} \quad (12)$$

$$\hat{\alpha} = \exp\left\{\frac{\sum_{i=1}^{n}\ln(t_i)}{n} - \frac{1}{n\beta}\sum_{i=1}^{n}\ln\left[-\ln(1-F)(t_i)\right]\right\}, \quad (13)$$

where, $F(t_i)$ is Bernard's median rank given as:

$$F(t_i) = \frac{i-0.3}{n+0.4}, \quad (14)$$

with n = number of sample used and also i = 1; 2,…, n is the ordered number of items

Prior Distribution and the Loss Function In Bayesian analysis the choice of prior is inevitable. The choice of a prior, most often than not depends on ones subjective knowledge and beliefs. In a case where there is enough evidence on the part of the researcher then an informative prior can be chosen otherwise a non-informative prior will be more appropriate. Consideration is given to Jeffreys non-informative prior for both parameters in this study which is obtained through Fisher information. Jeffreys' prior for the two parameters after employing and analysing the Fisher information matrix is given as,

$$u(\alpha, \beta) \propto \alpha^{-1}\beta^{-1}. \quad (15)$$

Having taking into account our prior information which is essential, the Bayes estimator shall be considered under two loss functions which is also indisputable in Bayesian estimation. The first loss function under consideration is, squared error loss function (SELF) which is symmetrical in nature. It gives equal opportunities to both under-estimation and over-estimation. The second loss function is linear exponential (LINEX) loss function, which was introduced by Varian 1975. It is asymmetrical in nature and takes into consideration overestimation and underestimation in which one of them could be dangerous as compared to the other depending on the situation or the problem at hand. At a point where the LINEX loss function approaches zero (0) then it turns to be approximately the same as SELF.

The squared error loss function is given as,

$$L(\alpha,\theta) = (\alpha - \theta)^2 \quad (16)$$

where, L(a, θ) is the loss incurred by adopting action a when the true state of nature is θ.

The linex loss function is also given as,

$$L(\Delta) = e^{a\Delta} - a\theta - 1; a \neq 0 \quad (17)$$

where, $\Delta = (\hat{\theta} - \theta)$ with $\hat{\theta}$ being the estimate of θ.

According to (Zellner 1986) the posterior expectation of the LINEX loss function is

$$E_\theta L(\Delta) = b\left[e^{a\hat{\theta}}E_\theta e^{-a\theta} - a(\hat{\theta} - E_\theta\theta) - 1\right] \quad (18)$$

The value of $\hat{\theta}$ that minimizes the above equation is,

$$\hat{\theta}_{BL} = -\frac{1}{a}\ln\left(E_\theta e^{-a\theta}\right) \quad (19)$$

provided $E_\theta(e^{-a\theta})$ exist and is finite.

BAYESIAN ESTIMATION

Squared Error Loss Function Bayes estimator $\hat{u}_{BS}$ of a function u = (α, β) of the unknown parameters α and β under SELF-Bayes estimator is the posterior mean. Therefore the SELF-Bayes estimator, according to (Lye et al. 1993) is given by:

$$\hat{u}_{BS} = (\alpha,\beta|t) = \frac{\int\int u(\alpha,\beta)\pi^*(\alpha,\beta)d\alpha d\beta}{\int\int \pi^*(\alpha,\beta)d\alpha d\beta} \quad (20)$$

It may be observed from (20) that the posterior distribution of (α,β) takes a ratio form that involves integration in both the numerator and denominator which cannot be reduced to a close form, therefore Lindley approximation is employed.

Lindley Procedure According to Lye et al. 1993, the posterior SELF-Bayes estimator of an arbitrary function u(θ) given by Lindley (1980) is:

$$E\{(\theta)|x\} = \frac{\int u(\theta)v(\theta)\exp[L(\theta)]d\theta}{\int v(\theta)\exp[L(\theta)]d\theta} \quad (21)$$

where, L(θ) is the log-likelihood and u(θ) v(θ) are arbitrary functions of θ. In applying this procedure, it is assumed that v(θ) is the prior distribution and u(θ) being some function of interest.

This can be approximated asymptotically by the following:

$$E\{u(\theta)|x\} = \left[u + \frac{1}{2}\sum_i\sum_j(u_{ij} + 2u_i\rho_j).\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}.\sigma_{ij}.\sigma_{kl}.u_l\right] \quad (22)$$

where, i; j; k; l = 1; 2,…, n; $\theta = (\theta_1; \theta_2,\ldots, \theta_m)$, and L is the log-likelihood function in (5).

From (22) the followings are obtained:

$$\rho 1 = -\frac{1}{\alpha}, \rho 2 = -\frac{1}{\beta}$$

$$u = \alpha, u_1 = \frac{\partial u}{\partial \alpha} = 1, u_{11} = u_2 = u_{22} = 0$$

$$u = \beta, u_2 = \frac{\partial u}{\partial \beta} = 1, u_1 = u_{11} = u_{22} = 0$$

$$L_{20} = \frac{\partial^2 L}{\partial \alpha^2} = -\frac{n}{\alpha^2}$$

$$L_{02} = \frac{\partial^2 L}{\partial \beta^2} = -\frac{n}{\beta^2} - \alpha \sum_{i=1}^{n} (t_i)^{\beta} \ln^2 (t_i)$$

$$\sigma_{11} = (-L_{20})^{-1}$$

$$\sigma_{22} = (-L_{02})^{-1}$$

$$L_{30} = \frac{\partial^3 L}{\partial \alpha^3} = \frac{2n}{\alpha^3}$$

$$L_{03} = \frac{\partial^3 L}{\partial \beta^3} = \frac{2n}{\beta^3} - \alpha \sum_{i=1}^{n} (t_i)^{\beta} \ln^3 (t_i)$$

Linear Exponential Loss Function The Bayes Estimator $\hat{\mu}_{BL}$ of a function u = u (exp(aα), exp(-aβ)under the LINEX loss function is given as:

$$\hat{\mu}_{BL} = \{[\exp(-a\alpha), \exp(-\alpha\beta)]; t\} = \frac{\int\int u[\exp(-a\alpha), \exp(-a\beta)]\pi^*(\alpha,\beta)\, d\alpha\, d\beta}{\int\int \pi^*(\alpha,\beta)\, d\alpha\, d\beta} \quad (23)$$

The LINEX loss is obtained by using the same Lindley procedure in (23) with:

$$u(\alpha) = e^{-a\alpha}\,;\; u(\beta) = e^{-a\beta}$$

$$u_1(\alpha) = e^{-a\alpha}\,;\; u_{11} = a^2 e^{-a\alpha}$$

$$u_2(\alpha) = e^{-a\beta}\,;\; u_{22} = a^2 e^{-a\beta}$$

SIMULATION STUDY

The estimators $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$ are maximum likelihood estimators of the parameters of the Weibull distribution, whiles $\hat{\alpha}_{LS}$ and $\hat{\beta}_{LS}$ are the least square estimators and $\hat{\alpha}_{BS}$ and $\hat{\beta}_{BS}$, $\hat{\alpha}_{BL}$ and $\hat{\beta}_{BL}$, are Bayesian estimators obtained. The estimated parameters are obtained based on simulation study. We considered a number of values for the parameters with different sample sizes. We chose a sample size of n = 25, 50 and 100 to take care of small, medium and large data sets. The different values of the parameters of the Weibull distribution considered are α = 0.5 and 1.5 and β = 0.8 and 1.2. The loss parameter is represented by a = ± 0.6 and ± 1.6. These were iterated 5000 times. Comparisons are made using mean squared error and conclusions given regarding the behaviour of the estimators.

TABLE 1. Mean squared error values for α and β with respect to MLE, LSE and Bayes

n	α	β	$\hat{\alpha}_{ML}$	$\hat{\beta}_{ML}$	$\hat{\alpha}_{LS}$	$\hat{\beta}_{LS}$	$\hat{\alpha}_{BS}$	$\hat{\beta}_{BS}$	$\hat{\alpha}_{BL}$ a = 0.6	$\hat{\beta}_{BL}$ a= [-0.6]	$\hat{\alpha}_{BL}$ a = 1.6	$\hat{\beta}_{BL}$ = [-1.6]
25	0.5	0.8	0.0179	0.023	0.0188	0.0269	0.0189	0.0253	0.0171	0.0215	0.0167	0.0186
		1.2	0.0075	0.0501	0.0078	0.0606	0.0078	0.0589	0.0074	0.0456	0.0074	0.0418
		0.8	0.0179	0.023	0.0188	0.0269	0.0189	0.0253	[0.018])	[0.0233]	[0.0195]	[0.0263]
		1.2	0.0075	0.0501	0.0078	0.0606	0.0078	0.0589	[0.0079]	[0.0558]	[0.0085]	[0.0620]
	1.5	0.8	0.1548	0.0212	0.1729	0.0271	0.1804	0.0207	0.1466	0.0201	0.1382	0.0210
		1.2	0.0689	0.0485	0.0747	0.0635	0.0798	0.0478	0.0641	0.0472	0.0609	0.0469
		0.8	0.1548	0.0212	0.1729	0.0271	0.1804	0.0207	[0.1699]	[0.0226]	[0.2074]	[0.0215]
		1.2	0.0689	0.0485	0.0747	0.0635	0.0798	0.0478	[0.0754]	[0.0495]	[0.0876]	[0.0488]
50	0.5	0.8	0.0089	0.0092	0.0094	0.0126	0.0091	0.0096	0.0089	0.0093	0.0084	0.0084
		1.2	0.0038	0.0204	0.0039	0.0294	0.0039	0.0223	0.0038	0.0192	0.0037	0.0194
		0.8	0.0089	0.0092	0.0094	0.0126	0.0091	0.0096	[0.0089]	[0.0094]	[0.0092]	[0.0102]
		1.2	0.0038	0.0204	0.0039	0.0294	0.0039	0.0223	[0.0039]	[0.0224]	[0.0039]	[0.0250]
	1.5	0.8	0.0796	0.0093	0.0861	0.0126	0.0859	0.0091	0.0761	0.0088	0.0716	0.0092
		1.2	0.0349	0.0212	0.0361	0.0287	0.0378	0.0209	0.0332	0.0209	0.0327	0.0199
		0.8	0.0796	0.0093	0.0861	0.0126	0.0859	0.0091	[0.0803]	[0.0092]	[0.0889]	[0.0095]
		1.2	0.0349	0.0212	0.0361	0.0287	0.0378	0.0209	[0.0368]	[0.0213]	[0.0385]	[0.0200]
100	0.5	0.8	0.0042	0.0042	0.0047	0.0066	0.0043	0.0043	0.0041	0.0042	0.0042	0.004
		1.2	0.0019	0.0094	0.002	0.0148	0.0019	0.0098	0.0019	0.0092	0.002	0.0093
		0.8	0.0042	0.0042	0.0047	0.0066	0.0043	0.0043	[0.0043]	[0.0044]	[0.0044]	[0.0044]
		1.2	0.0019	0.0094	0.002	0.0148	0.0019	0.0098	[0.0019]	[0.0099]	[0.0019]	[0.0101]
		0.8	0.0398	0.0043	0.0395	0.0065	0.0413	0.0043	0.0384	0.0041	0.0374	0.0042
		1.2	0.0174	0.0095	0.0177	0.0147	0.0181	0.0094	0.0168	0.0090	0.0168	0.0096
		0.8	0.0398	0.0043	0.0395	0.0065	0.0413	0.0043	[0.0409]	[0.0042]	[0.0184]	[0.0042]
		1.2	0.0174	0.0095	0.0177	0.0147	0.0181	0.0094	[0.0179]	[0.0098]	[0.0184]	[0.0099]

RESULTS AND DISCUSSION

DISCUSSION

The mean squared error values for the estimators estimates for alpha (α) and beta (β) are presented in Table 1, from which we observe that Bayesian estimator of the scale parameter under LINEX loss function with the loss parameter of a = 1.6 is shown to be better than all the other estimators except where the sample size is 100 with $\alpha = 0.5$ and $\beta = 1.2$ that maximum likelihood estimator gave the smallest MSE.

It indicates that the LINEX loss function has overestimated the scale parameter since the loss parameter at which the smallest MSE is observed is greater than zero. In the square brackets are the loss parameter of the mean squared error values of a = -0.6 and -1.6.

It has also been observed from Table 1, that the shape parameter (β) also known as the slope of the distribution from which one may be able to predict the physics behind the failure of a machine according to Abernethy (2006) is best estimated by Bayes under LINEX loss function since it has the smallest mean square error compared to maximum likelihood estimator, least square regression estimator and squared error loss function with respect to Bayes.

CONCLUSION

The estimated parameters of the Weibull failure time distribution from Bayesian estimator under LINEX loss function is comparatively the best with respect to maximum likelihood estimator, least square regression estimator and the squared error loss function under Bayes. When the sample size increases the mean squared error for all the estimators decreases in all cases.

REFERENCES

Abernethy, R.B. 2006. *The New Weibull Handbook*. 5th ed. Florida: Robert B. Abernethy.

Al-Athari, F.M. 2011. Parameter estimation for the double-pareto distribution. *J. Mathematics Statistics* 7: 289-294.

Al Omari, M.A. & Ibrahim, N.A. 2011. Bayesian survival estimator for weibull distribution with censored. *Data. J. Appl. Sci.* 11: 393-396.

Gibbons, I.D. & Vance, C.L. 1981. A simulation study of estimators for the 2-parameter Weibull distribution. *IEEE Tran. Reliabil.* R-30.

Lawless, J.F. 1982. *Statistical Models and Methods for Lifetime Data*. New York: Wiley.

Lindley, D.V. 1980. Approximate bayesian methods. *Trabajos Estadist* 31: 223-237.

Lye, L.M., Hapuarachchi, K.P. & Ryan, S. 1993. Bayes estimation of the extreme-value reliability function. *IEEE Trans. Reliab.* 42: 641-644.

Pandey, B.N., Dwividi, N. & Pulastya, B. 2011. Comparison between bayesian and maximum likelihood estimation of the scale parameter in weibull distribution with known shape under linex loss function. *J. Sci. Res.* 55: 163-172.

Preda, V., Panaitescu, E. & Constantinescu, A. 2010. Bayesian estimators of modified-weibull distribution parameters using Lindleys approximation. *WSEAS Transactions on Mathematics* 7: 539-549.

Syuan-Rong, H. & Shuo-Jye, W. 2011. Bayesian estimation and prediction for weibull model with progressive censoring. *J. Statistical Computation Simul.* 82: 1607-1620.

Zellner, A. 1986. Bayesian estimation and prediction using asymmetric loss functions. *J. American Statistical Association* 81: 446-451.

Zhang, L.F., Xie, M. & Tang, L.C. 2007. A study of two estimation approaches for parameters of weibull distribution based on WPP. *Reliab. Eng. Syst. Safety* 92: 360-368.

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